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THE WITT CLASSES OF TORSION LINKING FORMS OF $(4n-1)$-MANIFOLDS WITH PSEUDO-FREE CIRCLE ACTIONS

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In this paper we compute the Witt class of torsion linking forms of $(4n-1)$-dimensional closed oriented manifolds admitting orientation preserving pseudo-free circle actions.

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Pseudo-free circle action plumbing Witt groups
Chern class of a Seifert fibration Seifert manifold

1. Notations and Results

By a *pseudo-free circle action*, we mean a fixed point free action of the circle group G, consisting of complex numbers of absolute value 1, with the property that exceptional orbits are isolated. Throughout this paper, we shall be concerned with $(4n-1)$-dimensional smooth closed oriented manifolds with smooth effective orientation preserving pseudo-free circle actions.

Let X be such a manifold and E be the union of all exceptional orbits of X. Then $E^*=E/G$ is a finite subset of the orbit space X^* $(=X/G)$. If $E^*=\{x_1^*,\dots,x_s^*\}$, X is called a manifold with singularities $x_1^*,\dots,x_s^*$. Let m be the least common multiple of the orders of G_{x_j}, $j=1,\dots,s$, where G_{x_j} denotes the isotropy group at x_j. Factor by the C_m $(=\mathbb{Z}/m\mathbb{Z})$-action inside the circle action to get a principle G-bundle $\xi: X/C_m\to X^*$. Then X^* and X/C_m are rational cohomology manifolds even though they may not be manifolds.

Suppose $c_1(\xi)\in H^2(X^*:\mathbb{Q})$ is the Chern class of the corresponding complex line bundle. Then we call a cohomology class,

$$c_1=(1/m)c_1(\xi), \tag{1.1}$$

the *Chern class* of the Seifert fibration $X\to X^*$. Let $\operatorname{Ann}(c_1)=\{\alpha\in H^{2n-2}(X^*;\mathbb{Q})\mid \alpha\cup c_1=0\}$ and define a bilinear form μ_W on a vector space $W=H^{2n-2}(X^*;\mathbb{Q})/\operatorname{Ann}(c_1)$ by

$$\mu_W(\alpha,\beta)=\langle c_1\cup\bar\alpha\cup\bar\beta,[X^*]\rangle\in\mathbb{Q},\quad \bar\alpha,\bar\beta\in W. \tag{1.2}$$

Then μ_W is a symmetric nonsingular bilinear form, which depends not only on X but also on a given circle action θ on it. The Witt class represented by the rational form (W, μ_W) is called a *cohomology invariant* $w(G, X, \theta)$ of (G, X, θ).

Let C_a be a cyclic group of order a and let b be an integer relatively prime to a. Then $w(b/a)$ denotes an element of the Witt group $W(\mathbb{Q}/\mathbb{Z})$ of finite forms represented by (C_a, f), where $f(1,1) = b/a \in \mathbb{Q}/\mathbb{Z}$.

The main purpose of this paper is to prove the following two theorems:

1.3. Theorem. *Suppose X is a $(4n-1)$-manifold, $n \geqslant 2$, described above and $(\alpha_j; r_1^j, \ldots, r_{2n}^j)$ is an oriented slice invariant corresponding to a singularity x_j^* for $j = 1, \ldots, s$. Then the Witt class $Lk(X)$ of a torsion linking form of X is*

$$-w(r_1^1 \cdots r_{2n}^1/\alpha_1) - \cdots - w(r_1^s \cdots r_{2n}^s/\alpha_s) - \partial w(G, X, \theta),$$

where ∂ is the map in the Knebush sequence (1.6). Moreover, if the orientation is switched, then all negative signs in the formula are switched to positive signs.

We do not know much about the strange invariant $w(G, X, \theta)$. But the theorem tells us a relation between two invariants $Lk(M)$ and $w(G, X, \theta)$. Hence it can be used to compute one of them when the other is known.

By a *Seifert manifold*, we mean an oriented closed 3-manifold M admitting a fixed point free effective circle action. That is, a Seifert manifold is just the special case $n = 1$. Such a manifold is equivariantly classified by its *Seifert invariant* [9, 12]. In this paper, we shall use non-normalized Seifert invariants. They are described in Section 4. Our orientation conventions for Seifert manifolds is the same as that adopted in [6, 9, 11].

1.4. Theorem. *Let M be a Seifert manifold with associated Seifert invariant $(g\colon (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s))$ and suppose p/q is $e(M) = \sum_{j=1}^{s} (\beta_j/\alpha_j)$ expressed in lowest term. Then the Witt class of a torsion linking form of M is*

$$-w(\beta_1/\alpha_1) - \cdots - w(\beta_s/\alpha_s) - w(1/pq) \quad \text{if } e(M) \neq 0,$$

$$-w(\beta_1/\alpha_1) - \cdots - w(\beta_s/\alpha_s) \quad \text{if } e(M) = 0.$$

Since reversing the orientation of M replaces the Seifert invariant $(g\colon (\alpha_j, \beta_j))$ by $(g\colon (\alpha_j, -\beta_j))$ and hence replaces $1/pq$ by $-1/pq$, the Witt class of a torsion linking form of $-M$ is $w(\beta_1/\alpha_1) + \cdots + w(\beta_s/\alpha_s) + w(1/pq)$ or $w(\beta_1/\alpha_1) + \cdots + w(\beta_s/\alpha_s)$ according as $e(M) \neq 0$ or $e(M) = 0$.

The torsion linking form of a closed oriented $(4n-1)$-manifold X is a symmetric non-singular finite form defined by $\lambda(x, y) = \langle z \cup x, [X] \rangle \in \mathbb{Q}/\mathbb{Z}$, where $x, y \in \operatorname{Tor} H^{2n}(X\colon \mathbb{Z})$ and z is a pre-image of y under the Bockstein homomorphism $H^{2n-1}(X; \mathbb{Q}/\mathbb{Z}) \to H^{2n}(X; \mathbb{Z})$. The Witt class represented by $(\operatorname{Tor} H^{2n}(X; \mathbb{Z}), \lambda)$ is called the Witt class of a torsion linking form of X and it is denoted by $Lk(X) \in W(\mathbb{Q}/\mathbb{Z})$.

Suppose B is a closed oriented $4n$-manifold with $\partial B = X$ and let V be the image of the homomorphism $j^*: H^{2n}(B, \partial B; \mathbb{Q}) \to H^{2n}(B; \mathbb{Q})$. Then there exists a symmetric non-singular bilinear form $\mu_V : V \times V \to \mathbb{Q}$ defined by

$$\mu_V(j^*x, j^*y) = \langle x \cup j^*y, [B, \partial B]\rangle \in \mathbb{Q}.$$

This form represents an element $w(B)$ of the Witt group $W(\mathbb{Q})$ of rational forms. It follows from [1] that if an orientation of B induces the positive orientation of ∂B then we have

$$Lk(\partial B) = -\partial(w(B)). \tag{1.5}$$

Here ∂ is the map in the Knebush sequence,

$$0 \to W(\mathbb{Z}) \to W(\mathbb{Q}) \xrightarrow{\partial} W(\mathbb{Q}/\mathbb{Z}) \to 0. \tag{1.6}$$

Let $\pi: X \to X/G$ be the Seifert fibration. Then the mapping cylinder $M(\pi)$ of π may not be a manifold in general. Cutting out the small neighborhoods of the singularities $x_1^*, \ldots, x_s^*$ from $M(\pi)$, we obtain a closed oriented $4n$-manifold B. In Section 2, it will be shown that the boundary of B is a disjoint union of X and several $(4n-1)$-dimensional lens spaces. It will be shown in Section 3 that $w(B)$ is actually the invariant $w(G, X, \theta)$. By assembling these results and by using (1.5), we give the proof of Theorem 1.3.

Although we can prove Theorem 1.4 by using similar techniques to those used in the proof of Theorem 1.3, regarding a Seifert manifold as a boundary of a plumbed manifold, we do have a shorter proof (the author is indebted to Professor Walter D. Neumann for pointing out this alternative proof). In Section 4, we give the proof of Theorem 1.4 and have some examples supporting the formula in the theorem.

Finally, we prove that any Witt class of a finite form can be realized by a Seifert manifold in Section 5.

If Y is an oriented manifold with an action of a group G and Y/G is an oriented manifold, orient Y/G so that the orientation of Y/G followed by the natural orientation of the orbits gives the orientation of Y. Orient the boundary of an oriented manifold Y so that an orientation of ∂Y followed by an inward normal vector gives the orientation of Y.

All actions are assumed to be smooth and effective. Manifolds are also assumed to be smooth unless it is said to be the contrary.

2. The $4n$-manifold B and oriented slice invariants

Throughout this and the next section, X denotes a closed oriented $(4n-1)$-manifold $(n \geq 2)$ supporting a pseudo-free action of a circle group G and $x_1^*, \ldots, x_s^*$ are its singularities. Then $X^* - \{x_1^*, \ldots, x_s^*\}$ is a $(4n-2)$-manifold even though X^* may not be.

If x^* is one of the singularities, then an invariant tubular neighborhood of an exceptional orbit through x is topologically equivalent to $(G, G\times_{C_\alpha} D^{4n-2})$ where $C_\alpha = G_x \subset G$ and the action of C_α on $G=\{z_1 \mid z_1\bar z_1 = 1\}$ and $D^{4n-2} = \{(z_2,\dots,z_{2n}) \mid z_2\bar z_2 + \cdots + z_{2n}\bar z_{2n} \le 1\}$ is given as follows:

$$\begin{aligned}&\lambda \times z_1 \to \lambda^{r_1} z_1, \quad \lambda = \exp(2\pi i/\alpha),\\ &\lambda \times (z_2,\dots,z_{2n}) \to \rho(\lambda)(z_2,\dots,z_{2n}).\end{aligned} \tag{2.1}$$

Here $\rho: C_\alpha \to SO(4n-2)$ is a slice representation and it is faithful, since the action is effective and G is abelian.

It follows from a maximal torus theorem that there exists an element $h \in SO(4n-2)$ such that $h\rho(\lambda)h^{-1} \in SO(2)\times\cdots\times SO(2) \subset SO(4n-2)$. Define a diagonal C_α-action on $G\times D^{4n-2}$ as follows:

$$\begin{aligned}&\lambda \times z_1 \to \lambda^{r_1} z_1,\\ &\lambda \times (z_2,\dots,z_{2n}) \to h\rho(\lambda)h^{-1}(z_2,\dots,z_{2n}).\end{aligned} \tag{2.2}$$

Then a map ϕ of $G\times D^{4n-2}$ onto itself defined by $\phi(z_1,(z_2,\dots,z_{2n})) = (z_1, h(z_2,\dots,z_{2n}))$ induces an equivariant homeomorphism ϕ^* of $(G,(G\times D^{4n-2})/C_\alpha$ via (2.1)) onto $(G,(G\times D^{4n-2})/C_\alpha$ via (2.2)). Hence we can assume that $\rho(\lambda) = (\lambda^{r_2}, \lambda^{r_3},\dots,\lambda^{r_{2n}})$. We call the tuple of integers $(\alpha; r_1, r_2,\dots,r_{2n})$ the *oriented slice invariant* of a singularity x^* (we may normalize so that $r_1 = 1$, but in this paper we use non-normalized slice invariants). $\gcd(r_i,\alpha) = 1$, for $i = 1, 2, \dots, 2n$, since exceptional orbits are isolated.

2.3. Consider the diagonal action of C_α restricted to $\partial(G\times D^{4n-2}) = G\times S^{4n-3}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}(G, G\times S^{4n-3}, C_\alpha) & \xrightarrow[\mu]{/C_\alpha} & (G, G\times_{C_\alpha} S^{4n-3})\\ \Big\downarrow{\scriptstyle \nu\,|\,G\backslash} & & \Big\downarrow{\scriptstyle \nu'\,|\,G\backslash}\\ (S^{4n-3}, C_\alpha) & \xrightarrow[\mu']{/C_\alpha} & S^{4n-3}/C_\alpha\end{array}$$

Hence

$f = (\mathrm{id}\times\mu) \amalg \mu' : [0,1]\times(G\times S^{4n-3}) \amalg S^{4n-3} \to [0,1]\times(G\times_{C_\alpha} S^{4n-3}) \amalg (S^{4n-3}/C_\alpha)$ induces a continuous map $\tilde f: M(\nu) = D^2\times S^{4n-3} \to M(\nu')$, where $M(\nu)$ and $M(\nu')$ are the mapping cylinders of ν and ν', respectively.

Since $\tilde f^{-1}([t, z_1, (z_2,\dots,z_{2n})])$ is an orbit through $(tz_1, z_2,\dots,z_{2n})$ under the diagonal action of C_α on $D^2\times S^{4n-3}$, f induces a homeomorphism of $D^2\times_{C_\alpha} S^{4n-3}$ onto $M(\nu')$.

2.4. By abusing notations, ν' denotes the orbit map

$$G\times_{C_\alpha} D^{4n-2} \xrightarrow{G\backslash} G\backslash(G\times_{C_\alpha} D^{4n-2}) \approx D^{4n-2}/C_\alpha.$$

Then $D^{4n-2}/C_\alpha \subset X^*$ is a cone over S^{4n-3}/C_α with a singularity x^* as a vertex.

Let E be a subset $[0, 1/2]\times(G\times\mathrm{int}(D^{4n-2}))\coprod(\mathrm{int}(D^{4n-2}))/\approx$ of the mapping cylinder of ν'. Then E is actually a small neighborhood of a singularity x^* in $M(\pi)$, the mapping cylinder of the Seifert fibration $X\to X^*$. We claim that the boundary component $\partial\bar{E}$ created by cutting out E from $M(\pi)$ is a lens space $L^{4n-1}(\alpha; r_1, r_2, \ldots, r_{2n})$:

Define an action of the cyclic group C_α on $D^2\times D^{4n-2}=\{(z_1,(z_2,\ldots,z_{2n}))\mid z_1\bar{z}_1\leq 1,\ z_2\bar{z}_2+\cdots+z_{2n}\bar{z}_{2n}\leq 1\}$ by

$$\lambda\times(z_1,(z_2,\ldots,z_{2n}))\to(\lambda^{r_1}z_1, \lambda^{r_2}z_2,\ldots,\lambda^{r_{2n}}z_{2n}),$$

$\lambda=\exp(2\pi i/\alpha)$. Then the quotient space $\partial(D^2\times D^{4n-2})/C_\alpha$ is a lens space $L^{4n-1}(\alpha; r_1,\ldots,r_{2n})$. Now $\partial(D^2\times D^{4n-2})/C_\alpha$ is actually the space $(S^1\times_{C_\alpha}D^{4n-2})\cup(D^2\times_{C_\alpha}S^{4n-3})$ attached along their boundaries $S^1\times_{C_\alpha}S^{4n-3}$. The boundary component $\partial\bar{E}$ created by removing E from $M(\pi)$ can be regarded as $(G\times_{C_\alpha}D^{4n-2})\cup M(\nu')$ attached along their boundaries. We have shown in (2.3) that $M(\nu')$ is homoemorphic to $D^2\times_{C_\alpha}S^{4n-3}$. Hence the boundary component $\partial\bar{E}$ is a lens space $L^{4n-1}(\alpha; r_1,\ldots,r_{2n})$.

We summarize these facts in the following.

2.5. Lemma. *Let $(\alpha_j; r_1^j, r_2^j,\ldots,r_{2n}^j)$ be an oriented slice invariant corresponding to a singularity x_j^*, $j=1,2,\ldots,s$. Suppose $M(\pi)$ denotes the mapping cylinder of a Seifert fibration $X\to X^*$. Then $B=M(\pi)-\bigcup\{E_j\mid j=1,\ldots,s\}$ is a compact $4n$-manifold and ∂B is a disjoint union of X and lens spaces $L^{4n-1}(\alpha_j; r_1^j,\ldots,r_{2n}^j)$, $j=1,2,\ldots,s$. Here E_j is a neighborhood of x_j^* constructed above.*

2.6. Let $N^k=M(\pi)-\bigcup\{E_j\mid 1\leq j\leq k\}$. Then the Mayer–Vietoris sequence with rational coefficients for (N^1, E_1) gives rise to

$$0\to H_{4n}(N^1)\to H_{4n}(M(\pi))\to\mathbb{Q}$$
$$\to H_{4n-1}(N^1)\to H_{4n-1}(M(\pi))\to 0.$$

Since $M(\pi)\simeq X^*$, we have $H_{4n}(M(\pi))=H_{4n-1}(M(\pi))=0$. Successive applications of the Mayer–Vietoris sequence for (N^k, E_k), $k=1,2,\ldots,s$ yield $H_{4n}(B;\mathbb{Q})=0$, $H_{4n-1}(B;\mathbb{Q})=Q^s$, $H_{2n}(B;\mathbb{Q})=H_{2n}(X^*;\mathbb{Q})$. Furthermore, from the homology exact sequence for $(B,\partial B)$, we have an exact sequence

$$0\to H_{4n}(B,\partial B;\mathbb{Q})\to\mathbb{Q}^{s+1}\to\mathbb{Q}^s\to,$$

which implies $H_{4n}(B,\partial B;\mathbb{Z})=\mathbb{Z}$. Hence B is a compact oriented $4n$-manifold.

3. Proof of Theorem 1.3

3.1. Let X be such a manifold described in Section 1; $x_1^*,\ldots,x_s^*$ are the singularities of X and a cyclic group C_{α_j} of order α_j is the isotropy group at x_j. Suppose m is the least common multiple of $\alpha_1, \dots, \alpha_s$. Then, factoring by the C_m-action inside the pseudo-free circle group, we have the following commutative diagram of maps:

$$\begin{tikzcd} X \arrow[rr, "/C_m", "f"'] \arrow[dr, "/G"', "\pi"] & & \tilde{X} \arrow[dl, "\tilde{\pi}"', "/G"] \\ & X^* & \end{tikzcd}$$

Recall that $M(\pi)$ and $M(\tilde{\pi})$ denote the mapping cylinder of π and $\tilde{\pi}$ respectively and $\tilde{X}$ and X^* are rational cohomology manifolds. Since $X \to X^*$ is a principal circle bundle, there exists $U \in H^2(M(\tilde{\pi}), \tilde{X}; \mathbb{Q})$ (Thom class) such that

$$\Phi^*: H^*(X^*; \mathbb{Q}) \xrightarrow{\tilde{\pi}^*} H^*(M(\tilde{\pi}); \mathbb{Q}) \xrightarrow{\cup U} H^*(M(\tilde{\pi}), \tilde{X}; \mathbb{Q})$$

and

$$\Phi_*: H_*(M(\tilde{\pi}), \tilde{X}; \mathbb{Q}) \xrightarrow{\cap U} H_*(M(\tilde{\pi}); \mathbb{Q}) \xrightarrow{\tilde{\pi}_*} H_*(X^*; \mathbb{Q})$$

are isomorphisms. The commutative diagram implies that a map $(f \times \mathrm{id}) \coprod \mathrm{id}: (X \times [0, 1]) \coprod X^* \to (\tilde{X} \times [0, 1]) \coprod X^*$ induces a well-defined continuous map $F: M(\pi) \to M(\tilde{\pi})$.

3.2. Since $f_*: H_{4n-1}(X; \mathbb{Q}) \to H_{4n-1}(\tilde{X}; \mathbb{Q})$ is a multiplication by m, it follows from the homology ladder of $F: (M(\pi), X) \to (M(\tilde{\pi}), \tilde{X})$ that an induced homomorphism $F_*: H_{4n}(M(\pi), X: \mathbb{Q}) \to H_{4n}(M(\tilde{\pi}), \tilde{X}; \mathbb{Q})$ is also a multiplication by m.

Suppose $[X] \in H_{4n-1}(X; \mathbb{Z}) \otimes \mathbb{Q}$ be a fundamental class of an oriented manifold X. The cohomology manifolds $\tilde{X}$ and X^* are oriented by the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & H_{4n}(M(\pi), X: \mathbb{Q}) \arrow[r, "\partial"] \arrow[d, "F_*"] & H_{4n-1}(X; \mathbb{Q}) \arrow[r] \arrow[d, "f_*", "\approx"'] & 0 \\ 0 \arrow[r] & H_{4n}(M(\tilde{\pi}), \tilde{X}; \mathbb{Q}) \arrow[r, "\tilde{\partial}"] \arrow[d, "\Phi_*"] & H_{4n-1}(\tilde{X}; \mathbb{Q}) \arrow[r] & 0 \\ & H_{4n-2}(X^*; \mathbb{Q}) & & \end{tikzcd}$$

That is, $(1/m)F_*([M(\pi), X]) = \sigma_{4n}$, $\Phi_*(\sigma_{4n})$ and $(1/m)f_*([X])$ are fundamental classes of $M(\tilde{\pi})$, X^* and $\tilde{X}$ respectively, where $\partial([M(\pi), X]) = [X]$.

3.3. The following diagram is commutative:

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H^{2n}(M(\tilde{\pi}), \tilde{X}; \mathbb{Q}) \arrow[r, "j^*"] & H^{2n}(M(\pi); \mathbb{Q}) \arrow[r, dashed, "i^*"] & {} \\
& H^{2(n-1)}(M(\tilde{\pi}); \mathbb{Q}) \arrow[u, "\cup U"'] & & \\
& H^{2(n-1)}(X^*; \mathbb{Q}) \arrow[u, "\tilde{\pi}^*"'] \arrow[r, "c_1(\xi)"] & H^{2m}(X^*; \mathbb{Q}) \arrow[uu, "\approx", "\tilde{\pi}^*"'] &
\end{tikzcd}$$

Here $c_1(\xi)$ is the Chern class of the principal circle bundle $\xi: X \to X^*$. In fact,

$$\tilde{\pi}^*(\beta \cup c_1(\xi)) = \tilde{\pi}^*(\beta \cup \tilde{\pi}^{*-1}(j^* U)) = \tilde{\pi}^*(\beta) \cup j^* U = j^*(\tilde{\pi}^*(\beta) \cup U).$$

Recall $c_1 = (1/m)c_1(\xi)$ and $\mathrm{Ann}(c_1) = \{\beta \in H^{2(n-1)}(X^*; \mathbb{Q}) | \beta \cup c_1 = 0\}$. The commutative diagram implies that the vector space $W = H^{2(n-1)}(X^*; \mathbb{Q})/\mathrm{Ann}(c_1)$ is isomorphic onto $\operatorname{coker} \delta = \operatorname{im} j^* \equiv \tilde{V}$ by Thom isomorphism Φ^*. Furthermore, we have

$$\begin{aligned}
\mu_{\tilde{V}}(\Phi^*(\alpha), \Phi^*(\beta)) &= \langle j^*(\Phi^*(\alpha)) \cup \Phi^*(\beta), \sigma_{4n}\rangle \\
&= \varepsilon(\{j^*(\tilde{\pi}^*(\alpha) \cup U) \cup (\tilde{\pi}^*(\beta) \cup U)\} \cap \sigma_{4n}) \\
&= \varepsilon'(\{[j^* \tilde{\pi}^*(\alpha) \cup j^* U] \cup [\tilde{\pi}^*(\beta) \cup U]\} \cap \sigma_{4n}) \\
&= \varepsilon''((\alpha \cup \tilde{\pi}^{*-1}(j^* U) \cup \beta) \cap \Phi_*(U)) \\
&= \mu_W\langle \alpha \cup c_1(\xi) \cup \beta, [X^*]\rangle \\
&= m\mu_W\langle \alpha \cup c_1 \cup \beta, [X^*]\rangle.
\end{aligned}$$

Hence $(W, m\mu_W)$ and $(\tilde{V}, \mu_{\tilde{V}})$ are isometric.

Let E_j be a neighborhood of a singularity x_j^* described in (2.4) and $B = M(\pi) - \bigcup\{E_j | j = 1, \ldots, s\}$. Then we have shown in (2.6) that B is a compact oriented $4n$-manifold. Recall $w(B)$ denotes a Witt class represented by (V, μ_V), where V is the image of $j^*: H^{2n}(B, \partial B; \mathbb{Q}) \to H^{2n}(B; \mathbb{Q})$ and $\mu_V(\alpha, \beta) = \langle \alpha' \cup \beta, [B, \partial B]\rangle$, $j^*(\alpha') = \alpha$.

3.4. Lemma. $w(B) = w(G, X, \theta)$.

Proof. Consider the following cohomology ladder:

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H^{2n}(M(\tilde{\pi}), \tilde{X}; \mathbb{Q}) \arrow[r, "\tilde{j}^*"] \arrow[d, "F_2^*"'] & H^{2n}(M(\tilde{\pi}); \mathbb{Q}) \arrow[r, "\tilde{i}^*"] \arrow[d, "F_1^*"', "\approx"] & H^{2n}(\tilde{X}; \mathbb{Q}) \arrow[d, "f^*"'] \\
{} \arrow[r, "\delta"] & H^{2n}(M(\pi), X; \mathbb{Q}) \arrow[r, "j^*"] & H^{2n}(M(\pi); \mathbb{Q}) \arrow[r, "i^*"] & H^{2n}(X; \mathbb{Q})
\end{tikzcd}$$

Since there is a transfer map μ such that $\mu f^*(\beta) = (\text{order of } C_m)\beta$ (see [2]), f^* is a monomorphism. Moreover, F_1^* is an isomorphism, since $M(\tilde{\pi}) \simeq X^* \simeq M(\pi)$. Hence F_1^* isomorphically maps $\tilde{V} = \operatorname{im} \tilde{j}^*$ onto $V' = \operatorname{im} j^*$.

It was shown in (3.2) that $F_{2*}: H_{4n}(M(\pi), X; \mathbb{Q}) \to H_{4n}(M(\tilde{\pi}), \tilde{X}; \mathbb{Q})$ is a multiplication by m. Hence we have

$$\mu_{V'}(F_1^* \tilde{j}^*(\alpha), F_1^* \tilde{j}^*(\beta)) = m\mu_{\tilde{V}}(\tilde{j}^*(\alpha), \tilde{j}^*(\beta)). \tag{3.5}$$

In other words, $(V', \mu_{V'})$ is isometric to $(\tilde{V}, m\mu_{\tilde{V}})$.

By successive applications of Mayer–Vietoris cohomology sequences for (N^k, E_k), $k = 1, 2, \ldots, n$ (see (2.6)), we can show that $H^{2n}(M(\pi); \mathbb{Q}) \to H^{2n}(B; \mathbb{Q})$ induced by an inclusion is isomorphic. Let $\bar{E} = \bigcup \{\bar{E}_j \mid 1 \leq j \leq s\}$. Then we have a commutative diagram,

$$\begin{array}{ccccc}
H^{2m}(M(\pi), X; \mathbb{Q}) & \underset{\approx}{\xleftarrow{k^*}} & H^{2n}(M(\pi), X \cup \bar{E}; \mathbb{Q}) & \underset{\approx}{\xrightarrow{\text{excision}}} & H^{2n}(B, \partial B; \mathbb{Q}) \\
\downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j_1^*} & & \downarrow{\scriptstyle j_2^*} \\
H^{2n}(M(\pi); \mathbb{Q}) & \underset{\approx}{\longleftarrow} & H^{2n}(M(\pi); \mathbb{Q}) & \underset{\approx}{\longrightarrow} & H^{2n}(B; \mathbb{Q})
\end{array}$$

Hence we have $\operatorname{im} j^* = \operatorname{im} j_1^* = \operatorname{im} j_2^* = V$.

Let an orientation class of B be the image of the orientation class $[M(\pi), X]$ by the map ρ, where ρ is the composition of maps

$$H_{4n}(M(\pi), X : \mathbb{Q}) \underset{\approx}{\xrightarrow{k_*}} H_{4n}(M(\pi), X \cup \bar{E}; \mathbb{Q}) \underset{\approx}{\xleftarrow{\text{excision}}} H_{4n}(B, \partial B; \mathbb{Q}).$$

Then we have

$$\begin{aligned}
\mu_V(j_2^*\alpha, j_2^*\beta) &= \langle \alpha \cup j_2^*\beta, [B, \partial B] \rangle \\
&= \langle \alpha' \cup j^*\beta', [M(\pi), X] \rangle \\
&= \mu_{V'}(j^*\alpha', j^*\beta') \\
&= m\mu_{\tilde{V}}(F_1^{*-1} j^*\alpha', F_1^{*-1} j^*\beta') \quad \text{by (3.5).}
\end{aligned}$$

where $\alpha' = q^{-1}(\alpha)$, $\beta' = q^{-1}(\beta)$ and q is a composition of k^{*-1} and excision. On the other hand, it was shown in (3.3) that $(\tilde{V}, \mu_{\tilde{V}})$ is isometric to $(W, m\mu_W)$. Hence (V, μ_V) is isometric to $(W, m^2\mu_W)$ which implies $w(V, \mu_V) = w(W, \mu_W) = w(G, X, \theta) \in W(\mathbb{Q}/\mathbb{Z})$ □

3.5. Example. Define an action of a circle group G on $S^{4n-1} = \{(z_1, \ldots, z_{2n}) \in \mathbb{C}^{2n} \mid z_1\bar{z}_1 + \cdots + z_{2n}\bar{z}_{2n} = 1\}$ by $z \times (z_1, \ldots, z_{2n}) \xrightarrow{\theta} (zz_1, \ldots, zz_{2n})$, $z\bar{z} = 1$. Factor by the C_α-action inside the circle action to get a commutative diagram,

$$\begin{tikzcd} S^{4n-1} \arrow[rr, "/C_\alpha", "h"'] \arrow[dr] & & L^{4n-1}(\alpha; 1, 1, \ldots, 1) \arrow[dl] \\ & \mathbb{C}P^{2n-1} & \end{tikzcd}$$

Then it is not difficult to check $\mu_W(c_1(\gamma)^{n-1}, c_1(\gamma)^{n-1}) = -1$ (for example, one can use the remark in [4, p. 170] to see this). Here $c_1(\gamma)$ is the Chern class of the Hopf bundle $\gamma: S^{4n-1} \to \mathbb{C}P^{2n-1}$. Since h is a map of degree α, by using the results in (3.3) and (1.5), we can show that the Witt class of torsion linking form of $L^{4n-1}(\alpha; 1, 1, \ldots, 1)$ is $w(1/\alpha) \in W(\mathbb{Q}/\mathbb{Z})$. Furthermore, the invariant $w(G, S^{4n-1}, \theta)$ is actually a form over $\mathbb{Q}$ with matrix (-1) and hence Theorem 1.3 yields $Lk(S^{4n-1}) = 0$ (as it should be).

3.6. Remark. It was shown in [1] that $Lk(L^{4n-1}(\alpha; r_1, \ldots, r_{2n})) = w(r_1 \cdots r_{2n}/\alpha)$ under the hypotheses that α is an odd number. But their proof actually works for an arbitrary α.

Alternatively, we can also prove the same result by using the arguments in Example 3.5.

Proof of Theorem 1.3. By (2.5), ∂B is a disjoint union of X and lens spaces $L^{4n-1}(\alpha_j; r_1^j, \ldots, r_{2n}^j)$, $j = 1, 2, \ldots, s$. Hence it follows from (1.5) that

$$\partial w(B) = -\{Lk(X) + Lk(L^{4n-1}(\alpha_1; r_1^1, \ldots, r_{2n}^1)) + \cdots + Lk(L^{4n-1}(\alpha_s; r_1^s \cdots r_{2n}^s))\}.$$

By applying Lemma 3.4 and Remark 3.6, we obtain

$$Lk(X) = -\{w(r_1^1 \cdots r_{2n}^1/\alpha_1) + \cdots + w(r_1^s \cdots r_{2n}^s/\alpha_s)\} - \partial w(G, X, \theta),$$

and the proof is complete. □

4. Proof of Theorem 1.4.

A direct proof of Theorem 1.4 which can be done by modifying results and techniques in Sections 2 and 3 enables us to see geometrically how the singularities fit to the formula. But it is longer than even that of Theorem 1.3. In this section, we demonstrate an alternative proof of Theorem 1.4 by using the expression of a Seifert manifold via plumbing.

First of all we outline some known results which will be needed in this section (see [7] for further details). Let $M \to M/G$ be the Seifert fibration. Let $x_1^*, \ldots, x_s^*$ be the singularities of M. Let $T_1, \ldots, T_s$ be disjoint invariant tubular neighborhoods of orbits $G(x_1), \ldots, G(x_s)$ and $M_0 = M - \mathrm{int}(T_1 \cup T_2 \cup \cdots \cup T_s)$. Then $M_0 \to M_0/G$

is a principal circle bundle over a connected surface with non-empty boundary. Hence there exists a cross-section $S \subset M_0$. Let $S_j = S \cap \partial T_j$, then S_j is a curve in ∂T_j which is homologous in T_j to some multiple $\beta_j G(x_j)$ of the central curve of T_j. Let α_j be the order of the isotropy group G_{x_j}. Let g be the genus of the surface M/G. Then the *non-normalized Seifert invariant* is the collection of numbers

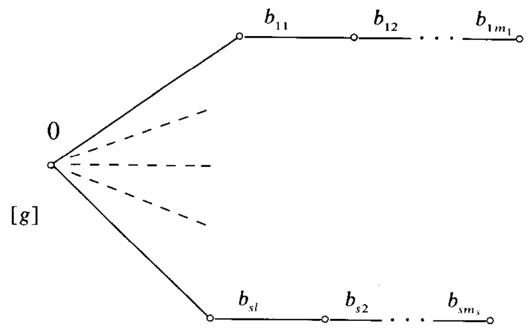
$$(g: (\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s)).$$

They satisfy $g \geq 0$, $\alpha_j \geq 1$, $\gcd(\alpha_j, \beta_j) = 1$. We permit β_j to be positive, negative, or zero (if $\alpha_j = 1$).

The Seifert invariant is not unique. The following operations are allowed; (i) permute the indices; (ii) add or delete a Seifert pair $(1, 0)$; (iii) replace (α_j, β_j), (α_k, β_k) by $(\alpha_j, \beta_j + m\alpha_j)$, $(\alpha_k, \beta_k - m\alpha_k)$ for some $m \in \mathbb{Z}$.

The number $e(M) = -\sum_{j=1}^{s} (\beta_j/\alpha_j)$, which is an invarant of the Seifert manifold by (iii), is called the *Euler number* of M. If M is a genuine circle bundle then $e(M)$ is the usual Euler number.

Expand α_j/β_j into a continued fraction $\alpha_j/\beta_j = b_{j1} - (b_{j2} - 1/(b_{j3} - \cdots - 1/b_{jm_j}) \cdots) = [b_{j1}, \dots, b_{jm_j}]$. Let $P(\Gamma)$ be a plumbed 4-manifold by the following weighted tree Γ:



The $[g]$ above means that the corresponding bundle being plumbed is the bundle of Euler number 0 over a surface genus g; all other bundles are bundles of Euler number b_{ij} over the sphere S^2.

4.1. Theorem ([7]). (i) *The matrix $A(\Gamma)$ of intersection form of the 4-manifold $P(\Gamma)$ can be diagonalized as*

$$\operatorname{diag}(e(M); d_{11}, \dots, d_{1m_1}; d_{21}, \dots, d_{2m_2}: \dots, d_{sm_s}),$$

with $d_{jk} = [b_{jk}, b_{jk+1}, \dots, b_{jm_j}]$.

(ii) *M is the boundary of $P(\Gamma)$.*

4.2. Suppose (V, f) is a rational form and $L \subset V$ is a free $\mathbb{Z}$-module such that $L \otimes \mathbb{Q} = V$ and $f(x, y) \in \mathbb{Z}$ for all $x, y \in L$. If $L^{\#}$ denotes the $\mathbb{Z}$-module $\{x \in V \mid f(x, y) \in \mathbb{Z}$ for all $y \in L\}$, then the map ∂: $W(\mathbb{Q}) \to W(\mathbb{Q}/\mathbb{Z})$ of (1.6) actually is a homomorphism defined by $\partial(V,f)=(L^{\#}/L,\bar{f})$, where $\bar{f}(x/L,y/L)=\rho f(x,y)\in\mathbb{Q}/\mathbb{Z}$, and ρ is the reduction $\mathbb{Q}\to\mathbb{Q}/Z$.

Let a and b be relatively prime non-zero integers and suppose (a/b) denotes the Witt class of the rational form represented by $(\mathbb{Q}\xi,f)$, where $f(\xi,\xi)=a/b$. Let $\zeta=b\zeta$ and $L=\{k\zeta \mid k\in\mathbb{Z}\}$. Then $L^{\#}$ is a $\mathbb{Z}$-module generated by $\{(k/ab)\zeta,\ k'\zeta \mid k,k'\in\mathbb{Z}\}$ and hence $L^{\#}/L=\{(k/ab)\zeta \mid k\equiv ab \pmod 1),\ k\in\mathbb{Z}\}$. On the other hand, $\bar{f}((1/ab)\zeta,(1/ab)\zeta)=(1/ab)^2 f(b\xi,b\zeta)=1/ab$.

$\mathbb{Z}/ab\mathbb{Z}=\mathbb{Z}/a\mathbb{Z}\oplus\mathbb{Z}/b\mathbb{Z}$ and the generators of the summands are $b(1/ab)\zeta$ and $a(1/ab)\zeta$. Moreover, $\bar{f}(b(1/ab)\zeta,\ b(1/ab)\zeta)=b^2(1/ab)=b/a$ and $\bar{f}(a(1/ab)\zeta,a(1/ab)\zeta)=a/b$. Hence the Witt classes of finite forms restricted to the summands are $w(a/b)$ and $w(b/a)$.

We summarize these facts in the following:

4.3. Lemma. (i) $\partial(b/a)=1/ab$.
(ii) $w(ab)=w(b/a)+w(a/b)$.

4.4. Lemma. *Let $[b_1,\ldots,b_m]=b_1-1/(b_2-\cdots-1/b_m)\cdots)$ be the continued fraction of α/β and $d_j=[b_j,\ldots,b_m]$ for $j=1,2,\ldots,m$. Then we have*

$$\partial((d_1)\oplus\cdots\oplus(d_m))=w(\beta/\alpha),$$

where (d_j) denotes the form over $\mathbb{Q}$ with matrix (d_j).

Proof.

$$\begin{aligned} d_1&=b_1-1/d_2=\alpha/\beta,\\ d_2&=b_2-1/d_3=(b_2d_3-1)/d_3,\\ &\vdots\\ d_j&=b_j-1/d_{j+1}=(b_jd_{j+1}-1)/d_{j+1},\\ &\vdots\\ d_m&=b_m. \end{aligned}$$

It follows from Lemma 4.3 that

$$\begin{aligned} &\partial(d_1)+\partial(d_2)+\cdots+\partial(d_j)+\cdots+\partial(d_m)=\\ &\quad=w(1/\alpha\beta)+w(1/(d_3(b_2d_3-1)))+\cdots+w(1/(d_{j+1}(b_jd_{j+1}-1))+\cdots\\ &\qquad+w(1/d_m)\\ &\quad=\{w(\beta/\alpha)+w(\alpha/\beta)\}+\{w(d_3/(b_2d_3-1))+w((b_2d_3-1)/d_3)\}+\cdots\\ &\qquad+\{w(d_{j+1}/(b_jd_{j+1}-1)+w((b_jd_{j+1}-1)/d_{j+1})\}+\cdots+w(1/d_m)\\ &\quad=w(\beta/\alpha)+\{w(\alpha/\beta)+w(1/d_2)\}+\cdots+\{w(b_j-1/d_{j+1})\\ &\qquad+w(1/d_{j+1})\}+\cdots+\{w(b_{m-1}-1/d_m)+w(1/d_m)\}\\ &\quad=w(\beta/\alpha).\quad\square \end{aligned}$$

Proof of Theorem 1.4. Since $P(\Gamma)$ is homotopic to the one point union of copies of the sphere S^2 and a surface of genus g, from the comology exact sequence of $(P(\Gamma), M)$ with rational coefficients, we have the following exact sequence:

$$0 \to \mathbb{Q}^{2g} \to \mathbb{Q}^{2g} \to H^2(P(\Gamma), M)$$

$$\xrightarrow{j^*} H^2(P(\Gamma)) \to \mathbb{Q}^{2g} \to \mathbb{Q}^{2g} \to 0.$$

Thus we have

$$\begin{aligned} Lk(M) &= -\partial A(\Gamma) \quad \text{(by 1.5)} \\ &= -\partial(e(M)) - \{\partial(d_{11}) + \cdots + \partial(d_{1m_1})\} - \cdots \\ &\quad -\{\partial(d_{s1}) + \cdots + \partial(d_{sm_s})\} \quad \text{(by Theorem 4.1)} \\ &= -w(1/pq) - w(\beta_1/\alpha_1) - \cdots - w(\beta_s/\alpha_s) \quad \text{(by Lemma 4.4).} \quad \square \end{aligned}$$

4.5. Example. Suppose M is a Seifert manifold with associated Seifert invariant $(g: (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s))$. Then by [12], we have $H_1(M; Z) = \mathbb{Z}^{2g} \oplus \langle Q_j, H \mid Q_1 + \cdots + Q_s = 0, \alpha_j Q_j + \beta_j H = 0\rangle$. Furthermore, if M is assumed to be a homology sphere then it was shown in [7] that the determinant of the presentation matrix of $H_1(M; \mathbb{Z})$ is $\pm(\alpha_1\alpha_2 \cdots \alpha_n)(\sum_{j=1}^{s} (\beta_j/\alpha_j)) = \pm 1$ and hence $\alpha_1, \alpha_2, \ldots, \alpha_s$ are pairwise coprime.

Suppose the determinant of the presentation matrix is $+1$. Then $e(M) = -1/(\alpha_1\alpha_2 \cdots \alpha_s)$. We thus have

$$Lk(M) = -\{w(\beta_1/\alpha_1) + \cdots + w(\beta_s/\alpha_s)\} - w(-1/(\alpha_1 \cdots \alpha_s)). \tag{4.5a}$$

Since $\alpha_1, \alpha_2, \cdots, \alpha_s$ are pairwise coprime, Lemma 4.3 gives rise to

$$\begin{aligned} w(1/(\alpha_1 \cdots \alpha_s)) &= w(\alpha_2 \cdots \alpha_s/\alpha_1) + w(\alpha_1\alpha_3 \cdots \alpha_s/\alpha_2) \\ &\quad + \cdots + w(\alpha_1\alpha_2 \cdots \alpha_{s-1}/\alpha_s). \end{aligned} \tag{4.5b}$$

By (4.5a) and (4.5b), we have

$$\begin{aligned} Lk(M) &= \sum_{j=1}^{s} \{-w(\beta_j/\alpha_j) + w(\alpha_1\alpha_2 \cdots \check{\alpha}_j \cdots \alpha_s/\alpha_j)\} \\ &= \sum_{j=1}^{s} \{w([-\beta_j + (\alpha_1 \cdots \check{\alpha}_j \cdots \alpha_s)]/\alpha_j) \\ &\quad + w([-\beta_j(\alpha_1 \cdots \check{\alpha}_j \cdots \alpha_s)/\alpha_j] \\ &\quad \times[-\beta_j + (\alpha_1 \cdots \check{\alpha}_j \cdots \alpha_s)]/\alpha_j)\} \end{aligned}$$

(by [3, p. 85]). Since

$$\begin{aligned} (\alpha_1 \cdots \alpha_s) \sum_{j=1}^{s} (\beta_j/\alpha_j) &= \beta_1(\alpha_2 \cdots \alpha_s) + \beta_2(\alpha_1\alpha_3 \cdots \alpha_s) + \cdots \\ &\quad + \beta_s(\alpha_1 \cdots \alpha_{s-1}) = 1, \end{aligned}$$

we have

$$\beta_j(\alpha_1 \cdots \check{\alpha}_j \cdots \alpha_s) \equiv 1 \pmod{\alpha_j} \quad \text{for each } j, \tag{4.5c}$$

and hence $Lk(M)=0$. Similarly, we can show that $Lk(M)$ is also zero for the case of $\det=-1$.

5. Realization of $W(\mathbb{Q}/\mathbb{Z})$

Let $(g\colon (\alpha_1,\beta_1),\ldots,(\alpha_k,\beta_k))$ be any tuple of integers with $g\geqslant 0$, $\alpha_j\geqslant 1$ for each j, and $\gcd(\alpha_j,\beta_j)=1$ for each j. Then we can always construct a Seifert manifold with associated Seifert invariant $(g\colon (\alpha_1,\beta_1),\ldots,(\alpha_k,\beta_k))$ (see [6, 9, 12]). As an application of Theorem 1.4, we obtain the following realization theorem.

5.1. Theorem *For each element $w(V,f)$ of $W(\mathbb{Q}/\mathbb{Z})$, there exists a Seifert manifold M such that $Lk(M)=w(V,f)$.*

Proof. It is well known [1, 3] that if $\mathbb{F}_p$ denotes the field with p elements, p a prime, then $W(\mathbb{Q}/\mathbb{Z})$ is isomorphic to $\bigoplus_p W(\mathbb{F}_p)$. Therefore, if $w(V,f)$ is an element of $W(\mathbb{Q}/\mathbb{Z})$ then $w(V,f)=w(\mathbb{Z}/p_1,f_1)+\cdots+w(\mathbb{Z}/p_k,f_k)$ for some prime numbers $p_1,\ldots,p_k$, where f_i is the rank one form defined by $f_i(1/p_i, 1/p_i)=b_i/p_i$ for some integer $b_i\leqslant p_i$.

Let u/v be $\sum_{i=1}^k (b_i/p_i)$ expressed in lowest term. We can always choose an integer n such that n^2 is relatively prime to v. Let $m=n^2-uv$. Then m is also relatively prime to v.

Note that p_i is relatively prime to b_i for each i. Suppose M is a Seifert manifold with associated Seifert invariant $(0\colon (p_1,-b_1),\ldots,(p_k,-b_k),(v^2,-m))$. Then the Euler number of M is

$$-\left\{\sum_{i=1}^k (-b_i/p_i)+(-m/v^2)\right\}=u/v+m/v^2=(m+uv)/v^2=n^2/v^2$$

and hence $Lk(M)=w(b_1/p_1)+\cdots+w(b_k/p_k)-w(1/(uv)^2)$. Since $w(1/(uv)^2)=0$ (note: a Witt class $w(\mathbb{Z}/m,f)=0$ if and only if m is a square), we have $Lk(M)=w(V,f)$. □

5.2. Remark. Suppose $w(V,f)$ is an element of $W(\mathbb{Q}/\mathbb{Z})$ such that $w(V,f)=w(b_1/p_1)+\cdots+w(b_k/p_k)$ and $b_i\equiv(p_1\cdots\check{p}_i\cdots p_k) \pmod{p_i}$ for each i, and let $m=q^2-(p_1\cdots p_k)$, where q is an integer that is relatively prime to $(p_1\cdots p_k)$. Then the Witt class of a torsion linking form of a Seifert manifold M with associated Seifert invariant $(0\colon ((p_1\cdots p_k),-1),\ldots,((p_1\cdots p_k)^2,-m))$ is

$$\begin{aligned}
&-w(-1/(p_1\cdots p_k))-w(-m/(p_1\cdots p_k)^2)-w(1/(p_1\cdots p_k q)^2)=\\
&\quad=w(p_2\cdots p_k/p_1)+\cdots+w(p_1\cdots p_{k-1}/p_k) \quad \text{by (4.5b)}\\
&\quad=w(b_1/p_1)+\cdots+w(b_k/p_k)=w(V,f).
\end{aligned}$$

In other words, $w(V,f)$ is the Witt class of a torsion linking form of a Seifert manifold with only two singular orbits of high degree of complexity. We believe that any element of $W(\mathbb{Q}/\mathbb{Z})$ could be realized by a Seifert manifold with a few singular orbits of high degree of complexity.

In concluding, we add the following question: Compute the Witt classes of torsion linking forms of closed oriented $(4n-1)$-manifolds admitting codimension two toral actions such that all isotropy groups are finite. In [10], a Seifert invariant of this kind of manifold was defined. Hence it is reasonable to expect that the Witt class $w(X, T^{4n-3})$ can be expressed in terms of the general Seifert invariant.

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